Distribution of persistent sites in diffusing systems

Damián H. Zanette
Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Río Negro, Argentina
(Received 10 September 1996)

The evolution of a diffusing field whose initial condition is given by a Gaussian random variable with zero mean is considered. It has been shown that, at time t, the number of sites where the field has not changed its initial sign decays as $n \sim t^{-\theta}$, where θ is a nontrivial exponent. Here the spatial distribution of these persistent sites in a one-dimensional system is numerically studied. It is shown that the two-point correlation function $C(x,t)$ decays, for small x, as $C(x,t) \sim x^{-2\theta}$. This power-law decay extends up to a typical length that increases as $x_c \sim t^{1/2}$, beyond which $C(x,t)$ is practically constant. As time elapses, the correlation function approaches a well-defined stationary state and, at any time, it collapses, if properly rescaled, to a universal curve that characterizes the form of the persistence domains. [S1063-651X(97)05503-7]

PACS number(s): 05.60.+w, 05.20.−y, 05.40.+j, 05.50.+q

The relevance of the diffusion equation

$$\partial_t \phi(x,t) = D \nabla^2 \phi(x,t) \tag{1}$$

to the description of a wide class of natural systems stands beyond any doubt. Because of its linearity, the general solution to this equation in the d-dimensional unbounded space,

$$\phi(x,t) = (4\pi D t)^{-d/2} \int d^d x \exp[-(x-x')^2/4Dt] \phi(x',0), \tag{2}$$

can be readily obtained. The spatiotemporal dependence in the kernel of this solution makes evident the well-known dynamical exponent of the diffusion process, $x^z \sim t$ with $z=2$. However, it has very recently been pointed out that, surprisingly enough, a second nontrivial exponent exists, associated with some dynamical aspects of this process [1,2].

Consider the diffusion process for an initial condition $\phi(x,0)$ given by a Gaussian stochastic variable with zero mean. As time elapses, the field $\phi(x,t)$ at a given point will approach its vanishing asymptotic value, typically changing its sign several times during the evolution. The number n of *persistent* sites, i.e., the sites where the initial sign of ϕ has remained unchanged along the evolution up to a given time t, is therefore expected to decay with t. It has been shown that this decay is algebraic, $n \sim t^{-\theta}$, with a nontrivial exponent that depends on the spatial dimension and on the space correlation properties of the initial condition.

In Ref. [1], this *persistence exponent* has been analytically derived within an approximated formalism based on the clever observation that the stochastic process defined by the normalized diffusive field at a given point x, $X(t) = \phi(x,t)/\langle \phi(x,t)^2 \rangle$, is stationary in the variable $T = \ln t$. There the relevance of these features to problems such as bimolecular reaction-diffusion processes has also been discussed. Other works [2,3] have pointed out that nontrivial persistent exponents occur also in problems related to diffusion, such as Glauber spin dynamics, where θ can be exactly calculated, and other nonequilibrium critical phenomena.

In this paper, we aim at characterizing the spatial distribution of persistent sites in a one-dimensional diffusion problem through the numerical calculation of the corresponding two-point correlation function. To begin with, let us define the *persistence index* $\pi(x,t)$ as

$$\pi(x,t) = \begin{cases} 1 & \text{if } \operatorname{sgn}\phi(x,t') = \operatorname{sgn}\phi(x,0) \quad \text{for all } t' \leq t \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Therefore, $\pi(x,0)=1$ for all x and, in addition, $\pi(x,t)^2 = \pi(x,t)$. According to the analytical and numerical results in [1], the mean value of the persistence index over the system decays, for long times, as a well-defined power of time

$$\langle \pi(x,t) \rangle \approx A t^{-\theta}. \tag{4}$$

Figure 1 shows the typical time evolution of $\langle \pi(x,t) \rangle$ from a

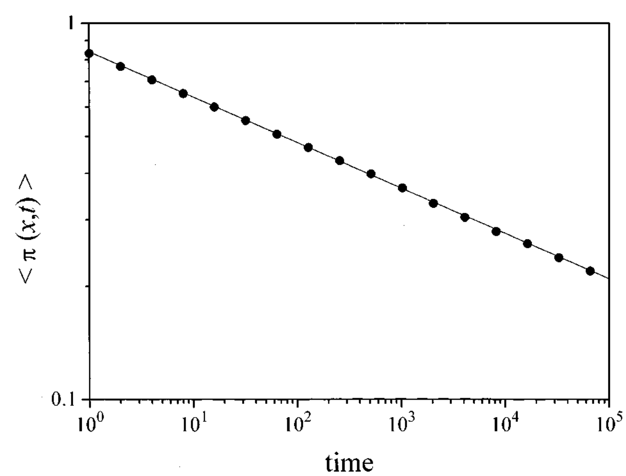

FIG. 1. Temporal evolution of the mean persistence index in a single realization over a 10^5-site one-dimensional lattice. The straight line in this log-log plot correponds to an exponent $\theta = 0.1207$.

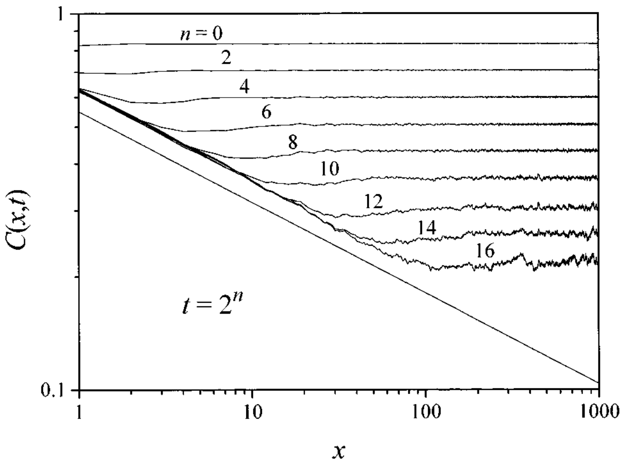

FIG. 2. Persistence correlation function vs position for several times, for the same realization as in Fig. 1. The straight line in this log-log plot corresponds to an exponent $\eta = 0.2414$.

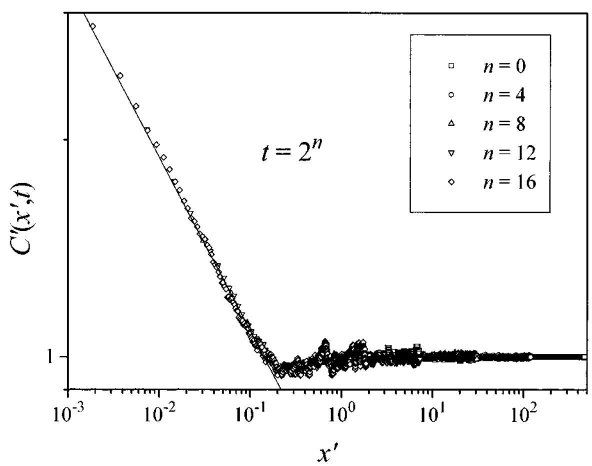

FIG. 3. Normalized correlation function vs normalized coordinate (see the text) for several times. Note the collapse to a single, well-defined curve. The straight line in this log-log plot corresponds to an exponent $\eta = 0.2414$. Its intersection with the line $C'(x',t) = 1$ defines the normalized crossover coordinate x'_c

numerical simulation performed as explained below. This case corresponds to a one-dimesional lattice, for which $\theta \approx 0.1207$.

The simplest characterization of the spatial distribution of persistent sites is given by the correlation function

$$C(x,t) = \frac{\int d^d y\, \pi(y,t)\, \pi(y+x,t)}{\int d^d y\, \pi(y,t)^2} = \frac{\langle \pi(y,t)\, \pi(x+y,t) \rangle}{\langle \pi(y,t) \rangle}. \tag{5}$$

This two-point correlation function satisfies $C(x,0) = 1$ for all x and $C(0,t) = 1$ for all t. For sufficiently large x, the persistence index is expected to be uncorrelated, so that $\langle \pi(y,t)\pi(y+x,t) \rangle \approx \langle \pi(y,t) \rangle \langle \pi(y+x,t) \rangle$ and

$$C(x,t) \approx \langle \pi(x,t) \rangle \approx A t^{-\theta}. \tag{6}$$

The crossover from the small-x decay of $C(x,t)$ to the large-x asymptotic value $At^{-\theta}$ defines a typical length $x_c(t)$, which characterizes the size of the *persistence domains*. Note that, since the only dynamical process that drives the system is simple diffusion, the crossover length $x_c(t)$ should scale with time with the usual exponent $x_c(t) \approx B t^{1/2}$.

Here we study the evolution of $C(x,t)$ from numerical simulations on a lattice. As suggested in [1], the diffusion equation (1) can be discretized, both in space and in time, as

$$\phi(x,t+\Delta t) = \phi(x,t) + a \sum_{\{y\}} [\phi(y,t) - \phi(x,t)], \tag{7}$$

with $a \equiv D\Delta t/\Delta x^2$ and where the sum runs over the nearest neighbors of site x. Taking $\Delta x \equiv 1$ and $\Delta t \equiv 1$, this numerical recursion scheme gives a particularly fast convergence of the persistence index to its asymptotic power-law decay for $a = 1/4d$, where d is the dimension of the system. We work on a one-dimensional 10^5-site lattice up to times of the order of 10^5 units.

Figure 2 shows the result for $C(x,t)$ as a function of x in a typical (single) realization for several values of t. It corresponds to the same realization that produced the data of Fig. 1. As advanced above, for moderately large times, $C(x,t)$ shows two well-defined regimes, which match at the crossover length $x_c(t)$. For $x > x_c$, the correlation is practically constant and can be seen to equal the mean value of the persistence index. On the other hand, for $x < x_c$, $C(x,t)$ decays from $C(0,t) = 1$ as x grows. At the crossover, the correlation seems to have a depletion zone, indicating the limit of the persistence domains.

The small-x spatial dependence of $C(x,t)$ exhibits two remarkable features. (i) The persistence correlation follows a power-law decay $C(x,t) \approx C x^{-\eta}$, even from the smallest values of x. (ii) This power-law decay is independent of time, so that it represents the large-time asymptotic value of $C(x,t)$ for every point x: $C(x,t) \to C x^{-\eta}$ for $t \to \infty$. The exponent η in the spatial decay of $C(x,t)$ is not independent of the persistence exponent θ, but they are connected through the dynamical exponent of diffusion. In fact, the crossover length $x_c(t)$ can be defined as the point of intersection of the two regimes of spatial dependence in $C(x,t)$:

$$C x_c^{-\eta} = A t^{-\theta}. \tag{8}$$

Taking into account that, as stated before, $x_c \approx B t^{1/2}$, we obtain

$$\eta = 2\theta \tag{9}$$

and $C \approx A B^{\eta}$. The straight line in Fig. 2 stands, in the log-log plot, for a power-law decay with exponent $\eta = 0.2414$, which corresponds to the value of θ for one-dimensional systems. The agreement with the numerical simulations supports our previous argument.

The scaling features discussed above suggest that the correlation profiles can collapse to a universal, time-independent curve if both $C(x,t)$ and x are conveniently scaled. Since, for $x < x_c$, $C(x,t) \propto x^{-\eta}$, the scaled quantities

$$C'(x',t) \equiv \frac{C(x,t)}{A t^{-\theta}}, \quad x' \equiv \frac{x}{(A t^{-\theta})^{-1/\eta}} \propto \frac{x}{t^{1/2}} \tag{10}$$

maintain the same power-law interdependence for small x'. For large x' instead, $C'(x',t)$ approaches unity. In Fig. 3, some of the numerical data of Fig. 2 have been scaled according to Eq. (10). Their collapse to a single curve is apparent. This curve provides then a statistical description of the persistence domains at any time. It shows the small-x power-law decay of the correlation, stressed in Fig. 3 by a straight line, which extends up to the scaled crossover length $x'_c \approx 0.13$. The asymptotic large-x value $C'(x',t) = 1$ is attained beyond the depletion zone mentioned above.

The recurrent appearance of power laws in the geometry and in the evolution of dynamical systems has been recognized as a clue to the complex interplay of the elements that constitute such systems. In fact, fractals [4], criticality [5], and algebraic time dependence [6] are typical features in the macroscopic dynamics of many coupled elements. The presence of a nontrivial power-law time dependence in a problem as simple as diffusion seems to indicate that coupled elements are able to display some kind of complexity even under linearity conditions.

The results presented here show how this kind of complexity reflects in the spatial statistics of the system, through the occurrence of a power-law decay in the correlation function of the persistence index. This power law, which, according to the previous discussion, is strongly linked by diffusion to the algebraic time decay of the persistent site number, is expected to appear in higher-order statistical features as well. Moreover, in view of the results obtained for the persistence index in higher-dimensional systems, it can be conjectured that the power-law dependence of the persistence correlation function is a generic characteristic of diffusion, independent of the spatial dimension.

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